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TIME OF TOTAL CREEP RUPTURE OF A BEAM UNDER COMBINED TENSION AND BENDING

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Abstract—Based upon two fundamental physical laws: Odqvist's creep law and Kachanov's brittle rupture law, the problem of total rupture of a beam of rectangular cross section under a bending moment and an axial tensile force has been considered. A power series method has been used to solve the basic integral equation and to perform numerical calculations.

NOTATION

$a(t)$	half of the height of the rectangular cross section at time t
a^*	height $a(t)$ for which $\sigma_x \geq 0$ in the whole section
b_0	half of the width of the rectangular cross section
m	material constant (in Kachanov's rupture law)
n	material constant (in Odqvist's creep law)
p	$= \frac{1}{q}$ non-dimensional parameter
q	$= \frac{\varkappa a}{\mu}$ non-dimensional parameter
s_{ij}	components of the deviator of stress
t	time
t_1	time of arising of first cracks
t_{1M}	time of arising of first cracks under pure bending
t_{1N}	time of arising of first cracks under pure tension
t_2	time of rupture of the cross section to the height a^*
t_z	time of total rupture
t_{zM}	time of total rupture under pure bending
x, y, z	dimensional coordinates
$\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{e}_i, \bar{f}_i, \bar{g}_i, a_i, d_i, e_i, f_i, g_i, s_i$	coefficients of power series
A	material constant (in Kachanov's rupture law)
F	area of the cross section of the bar
M	bending moment
N	axial tensile force
T	temperature
K_i, L_i, R_i, S_i, U_i	coefficients of power series
$C_1, C_2, D_1, D_2, E_1, E_2$	integration constants
$\alpha_i, \beta_i, \gamma_i, \nu_i$	coefficients of power series
δ	$= \frac{aN}{10M}$ parameter (inversion of the non-dimensional eccentricity)
$\dot{\varepsilon}_{ij}$	components of the strain rate tensor
$\dot{\varepsilon}_x$	rate of relative elongation change
η	$= \frac{y}{a}$ non-dimensional coordinate

η^*	$= -\frac{1}{q}$ coordinate determining the location of neutral axis
$\varkappa$	rate of bar axis curvature change
μ	rate of relative elongation of the bar axis
σ_x	normal stress
σ_e	stress intensity
σ_0	material constant (in Odqvist's creep law)
σ_c	$= \sigma_0\mu^{1/n}$ parameter
τ	apparent variable in time dimension
Ψ	scalar function characterizing continuity of material
Δ	$= \frac{10M}{aN}$ parameter (non-dimensional eccentricity)

1. INTRODUCTION

RUPTURE problems of solid bodies have recently been the subject of intensive investigations, both theoretical and experimental (see e.g. [5]). This interest is quite justified because a knowledge of the rupture time due to the action of such factors as, for example, load, temperature, aggressive medium, etc., permits the design of a safe structure.

Structures in an unfavourable environment, such as high temperature for metals, are especially exposed to rupture danger. The increased creep, occurring under these conditions, results in a gradual decohesion leading to a total rupture of the material. Of course, every structure, undergoes rupture when the external load intensity exceeds a certain limit value. In creep however, even a small load can cause a rupture after a certain period of time has elapsed and, thus, rupture under creep conditions should be especially carefully analysed.

The present paper considers the solution of a creep rupture problem under an uniaxial nonhomogeneous state of stress, realized in a prismatic bar of rectangular cross section by loading it with a bending moment M and an axial force N. The forced limitation upon the shape of the bar cross section restricts the generality of the considerations, but, on the other hand, simplifies it considerably, and it has no influence upon the qualitative solution.

In determining the stress distribution, a knowledge of which is indispensable for the analysis of the rupture process, we employ the formulae derived by S. Piechnik in his papers [17–19].

The first papers in the subject area of this paper were those by H. Hencky [6] and F. K. G. Odqvist [15]. Considerable progress was achieved by E. L. Robinson [21], L. M. Kachanov [9, 10, 12] and F. K. G. Odqvist [13, 16], in the field of brittle rupture and by N. J. Hoff [7, 8], in the area of ductile rupture.

Chrzanowski [1–3] considered the problem of creep rupture with the time of arising of the first cracks as an adopted criterion of destruction. Knowledge of the ratio t_z/t_1 (the time of total rupture to the time of arising of the first cracks) will make it possible to appreciate the correctness of this criterion.

The problem of propagation of the brittle rupture front was considered by Kachanov [11] for a prismatic bar loaded with a bending moment only. F. K. G. Odqvist [16] derived a differential equation determining the rupture front motion for simultaneous loading with an axial force and a bending moment; he did however not consider it further after indicating the mathematical difficulties connected with integration of this equation.

Approaching this problem in a somewhat different way we have succeeded to overcome these difficulties in the present paper.

2. BASIC PHYSICAL LAWS AND ASSUMPTIONS

According to Odqvist's scheme (Fig. 1) the creep process is assumed to be established from $t = 0$ up to the time of total rupture.

For an uniaxial state of stress the equation of state is assumed in the form [14]:

$$\sigma_x = \sigma_0 |\dot{\varepsilon}_x|^{(1-n)/n} \dot{\varepsilon}_x \tag{2.1}$$

where σ_0, n are material constants.

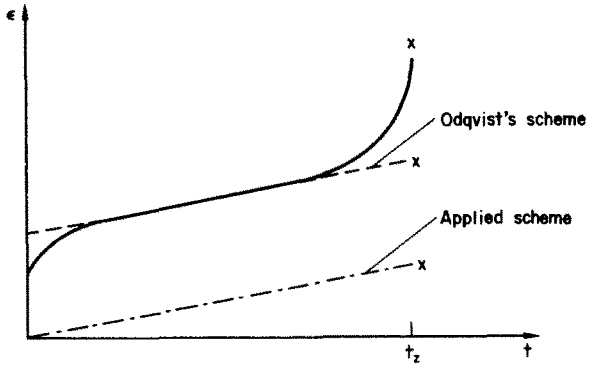

FIG. 1.

The rupture process is described by a theory given by Kachanov in Ref. [11]. The basic equation is:

$$\frac{d\Psi}{dt} = -A\left(\frac{\max \sigma_x}{\Psi}\right)^m \tag{2.2}$$

where A, m are material constants and Ψ is a function characterizing continuity of the material.

The equation governing the motion of the rupture front is:

$$A(m+1) \int_0^t \sigma_x^m(\tau)\, d\tau = 1. \tag{2.3}$$

The equation does not refer to a fixed point but is connected with a current point in which at the given t destruction occurs; thus, it describes the rupture front motion. The form (2.3) still requires certain elucidations which will be given in further parts of the present paper.

Among other assumptions adopted in the present paper, the following ones concern the physical side of the phenomenon:

— constancy of temperature,
— unchangeability of external load during the destruction process,
— isotropy of the material,
— equal properties for compression and tension.

In the geometry of the problem, the strains and the displacements are assumed to be small.

3. INITIAL EQUATIONS

As is seen from equations (2.2) and (2.3) the stress distribution must be known to determine both the time when the first cracks arise as well as the rupture front motion. This distribution—according to results of Ref. [19]—can be written in the form

$$\sigma_x = \sigma_c(1+q\eta)^{1/n} \quad \text{for} \quad q < 1$$

and for $q > 1$

$$\sigma_x = \begin{cases} \sigma_c(1+q\eta)^{1/n} & \text{for} \quad \eta \geq \eta^* \\ -\sigma_c(-1-q\eta)^{1/n} & \text{for} \quad \eta \leq \eta^* \end{cases} \tag{3.1}$$

where

$$q = \frac{\varkappa a}{\mu}, \qquad \sigma_c = \sigma_0 \mu^{1/n},$$

$\varkappa$ and μ are rates of change of curvature and relative elongation of the bar axis, respectively and $\eta = y/a$—non dimensional coordinate (see Fig. 2), $\eta^* = -1/q$—determines the situation of the neutral axis.

The unknown values σ_c and q are determined from the equilibrium conditions which are in integral form:

$$\begin{aligned} \int\int_F \sigma_x \, dF &= N \\ \int\int_F \sigma_x y \, dF &= M. \end{aligned} \tag{3.2}$$

This system of equations will be solved by describing σ_x by power series. This makes it necessary to consider the following two cases separately:

$q_0 < 1$—small eccentricity

$q_0 > 1$—large eccentricity.

In the sequel quantities at $t = 0$ will be denoted with the subscript zero.

A gradual destruction of the cross section and a decrease of its height—maintaining the point of application of the force N—will cause an increase of the bending moment, and thus of the eccentricity as well. Considering the case of a small eccentricity the fact should be taken into account that from a certain $a = a^*$ a change in the character of the stress distribution will take place; a total rupture will occur for the case of a large eccentricity. Because of it, the case of large eccentricity i.e. when at $t = 0$ $q_0 > 1$—will be considered first and next the obtained solution will be used to consider a small initial eccentricity.

4. LARGE ECCENTRICITY

4.1. *Distribution of stress*

For a large eccentricity the distribution of the normal stress has been shown in Fig. 2. Developing the expression $(1+q\eta)^{1/n}$ in power series the system (3.2) can be presented:

$$\begin{aligned} N &= 4\sigma_c q^{1/n} a^{(1+n)/n} b_0 \sum_{i=1}^{\infty} \beta_i p^{2i-1} \\ M &= 4a\sigma_c q^{1/n} a^{(1+n)/n} b_0 \sum_{i=0}^{\infty} \alpha_i p^{2i} \end{aligned} \tag{4.1}$$

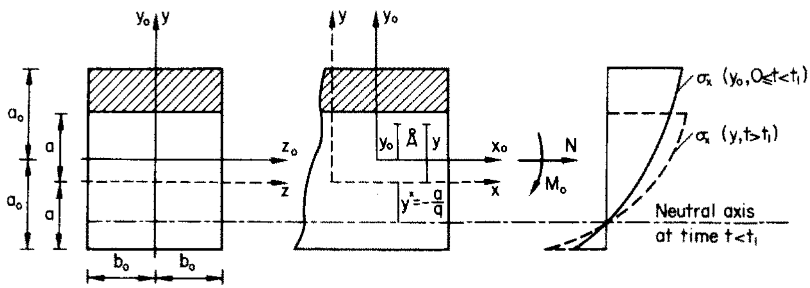

FIG. 2.

where

$$\beta_i = \begin{cases} 1 & \text{for } i = 1 \\ \dfrac{1}{(2i-1)!n^{2i-2}} \displaystyle\prod_{j=1}^{2i-3} (1-jn) & \text{for } i \geq 2 \end{cases}$$

$$\alpha_i = \begin{cases} \dfrac{n}{2n+1} & \text{for } i = 0 \\ \dfrac{1}{(2i)![1-2(i-1)n]n^{2i-1}} \displaystyle\prod_{j=1}^{2i-1} (1-jn) & \text{for } i \geq 1 \end{cases}$$

and $p = 1/q$.

Forming a quotient $(aN)/(10M)$† and performing division of series one obtains

$$\frac{aN}{10M} = \frac{aN}{10[M_0 + N(a_0 - a)]} \stackrel{\text{df}}{=} \delta(a) = \sum_{i=1}^{\infty} \gamma_i p^{2i-1} \tag{4.2}$$

where

$$\gamma_i = \frac{1}{10\alpha_0}\left[\beta_i - \sum_{j=1}^{i-1} \alpha_j \gamma_{i-j}\right].$$

† The multiplier $\frac{1}{10}$ was introduced to simplify numerical calculations.

After inversion of this series

$$p(a) = \sum_{i=1}^{\infty} v_i \delta^{2i-1}(a), \tag{4.3}$$

where

$$v_1 = \frac{1}{\gamma_1}, \qquad v_2 = -\frac{\gamma_2}{\gamma_1} v_1^3, \qquad v_3 = -\frac{1}{\gamma_1}(3\gamma_2 v_1^2 v_2 + \gamma_3 v_1^5), \ldots$$

the stress distribution can be determined as:

$$\sigma_x = \begin{cases} \dfrac{N}{4a^{(n+1)/n} b_0 \sum\limits_{i=1}^{\infty} \beta_i p^{2i-1}} (y+ap)^{1/n} & \text{for } -\dfrac{a}{q} < y \le a \\ \dfrac{-N}{4a^{(n+1)/n} b_0 \sum\limits_{i=1}^{\infty} \beta_i p^{2i-1}} (-y-ap)^{1/n} & \text{for } -a \le y < -\dfrac{a}{q} \end{cases} \tag{4.4.1}$$

or

$$\sigma_x = \begin{cases} \dfrac{M}{4a^{(2n+1)/n} b_0 \sum\limits_{i=0}^{\infty} \alpha_i p^{2i}} (y+ap)^{1/n} & \text{for } -\dfrac{a}{q} < y \le a \\ \dfrac{-M}{4a^{(2n+1)/n} b_0 \sum\limits_{i=0}^{\infty} \alpha_i p^{2i}} (-y-ap)^{1/n} & \text{for } -a \le y < -\dfrac{a}{q}. \end{cases} \tag{4.4.2}$$

The above formulae are valid when $p \le 1$ i.e. when $\delta(a) \le \delta_{\max}$.
For $t = 0$, thus, when $a = a_0$

$$\delta_{0\max} = \sum_{i=1}^{\infty} \gamma_i. \tag{4.5}$$

If after a time the section begins to crack then

$$\delta(a) = \frac{1}{10\left[\dfrac{M_0}{aN} + \dfrac{a_0 - a}{a}\right]}$$

begins to decrease [member $(a_0 - a)/a$ increases] and is permanently smaller than δ_0. Thus, the introduced formulae are valid for an arbitrary t if only for $t = 0$ we have

$$\frac{a_0 N}{10 M_0} \le \delta_{0\max}$$

4.2. *Time of arising of first cracks*

For $0 \leq t < t_1$ there is $a = a_0$, $\delta = \delta_0 = \text{const.}$, $p = p_0 = \text{const.}$, $y = y_0$, $M = M_0$, and so in this period of time the stress is independent of time. The rupture time is determined by:

$$t_1 = \frac{1}{A(m+1)(\max \sigma_x)^m},$$

which after making use of (4.4) takes the form

$$t_1 = \frac{\left[4a_0 b_0 \sum_{i=1}^{\infty} \beta_i p_0^{2i-1}\right]^m}{A(m+1)N^m(1+p_0)^{m/n}}, \tag{4.6}$$

or

$$t_1 = \frac{\left[4a_0^2 b_0 \sum_{i=0}^{\infty} \alpha_i p_0^{2i}\right]^m}{A(m+1)M_0^m(1+p_0)^{m/n}}. \tag{4.7}$$

In the particular case when $N = 0$ we have $p_0 = 0$, and thus

$$t_{1M} = \frac{[4a_0^2 b_0 \alpha_0]^m}{A(m+1)M_0^m}.$$

If we denote

$$4a_0^2 b_0 \alpha_0 = 4a_0^2 b_0 \frac{n}{2n+1} = \frac{2b_0}{1+\dfrac{1}{2n}} a_0^{2+1/n} a_0^{-1/n} \overset{\text{df}}{=} \frac{I_{m0}}{a_0^{1/n}}$$

so

$$t_{1MK} = \left[\frac{I_{m0}}{M_0}\right]^m \frac{1}{A(m+1)a_0^{m/n}}.$$

This formula is in agreement with that obtained by L. M. Kachanov [11].

Distribution of continuity at time $t = t_1$ for $y_0 > a_0 p_0$ (for tension stress) is given by the equation:

$$\Psi^m(y_0, t_1) = 1 - \left[\frac{\eta_0 + p_0}{1+p_0}\right]^{m/n}. \tag{4.8}$$

4.3. *Rupture time of the whole cross section*

For $t > t_1$ the integral of equation (2.2) takes the form

$$1 - \Psi^{m+1} = A(m+1)\int_0^t \sigma_x^m(\tau)\, d\tau \tag{4.9}$$

where

$$\sigma_x = \begin{cases} f_1[y(t)] & \text{for } 0 \le t < t_1 \\ f_2[y(t), \tau] & \text{for } t_1 < t \le t_z. \end{cases}$$

L. M. Kachanov suggested in his paper [11] the following notation for a similar problem (pure bending):

$$1-\Psi^{m+1} = A(m+1)\int_0^t f_2[y(t),\tau]\,\mathrm{d}\tau \tag{4.10.1}$$

which after substituting (4.4) for σ_x will take the following form for our problem:

$$1-\Psi^{m+1} = A(m+1)\left(\frac{N}{4b_0}\right)^m \int_0^t \frac{[y(\tau)-a(\tau)p(\tau)]^{m/n}}{\left[a^{(n+1)/n}(\tau)\sum_{i=1}^{\infty}\beta_i p^{2i-1}(\tau)\right]^m}\,\mathrm{d}\tau. \tag{4.10.2}$$

The two unknown functions $a(\tau)$ and $p(\tau)$ occurring here are connected with each other with dependence (4.2) and one of them e.g. $a(\tau)$ can be eliminated.

The notation (4.10.1) is correct only in context with further operations which will be carried out on this equation i.e. having been twice differentiated it will be converted into a differential equation. This transformation will be possible only under certain supplementary assumptions which will be discussed at a later time.

When the rupture front reaches the established point there is $y(t) = a(t)$ and $\Psi[a(t)] = 0$ and equation (4.9) takes the form:

$$1 = A(m+1)\left\{\int_0^{t_1} f_1[a(t)]\,\mathrm{d}\tau + \int_{t_1}^{t} f_2[a(t),\tau]\,\mathrm{d}\tau\right\} \tag{4.11.1}$$

and (4.10)—the form suggested by Kachanov:

$$1 = A(m+1)\int_0^t f_2[a(t),\tau]\,\mathrm{d}\tau. \tag{4.11.2}$$

It will be demonstrated that the above described transformations of the integral equations (4.11.1) and (4.11.2) will in both cases result in the same differential equation.

Let us first consider equation (4.11.1) and differentiate it with regard to time

$$0 = \int_0^{t_1}\frac{\partial f_1}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}\tau + \int_{t_1}^{t}\frac{\partial f_2}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}\tau + f_2[a(t),t].$$

So

$$\int_{t_1}^{t}\frac{\partial f_2}{\partial a}\mathrm{d}\tau = -\frac{1}{\mathrm{d}a/\mathrm{d}t}f_2[a(t),t] - \frac{\partial f_1}{\partial a}t_1$$

is obtained. After further differentiation we obtain:

$$0 = \frac{\partial^2 f_1}{\partial a^2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 t_1 + \frac{\partial f_1}{\partial a}\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}t_1 + \frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\int_{t_1}^{t}\frac{\partial f_2}{\partial a}\mathrm{d}\tau + \frac{\mathrm{d}a}{\mathrm{d}t}\int_{t_1}^{t}\frac{\partial^2 f_2}{\partial a^2}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}\tau + 2\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\partial f_2}{\partial a}.$$

Assuming that the functions $f_1[a(t)]$, $f_2[a(t),\tau]$ are linear with regard to $a(t)$, the expression containing second derivatives of these functions vanishes; after elimination of the integral and arranging it, the following is obtained:

$$2\frac{\partial f_2}{\partial a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2-\frac{\mathrm{d}^2a}{\mathrm{d}t^2}f_2[a(t),t] = 0.$$

Let us consider equation (4.11.2) and differentiate it with regard to time:

$$0 = \int_0^t \frac{\partial f_2}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}\tau+f_2[a(t),t].$$

The next differentiation yields:

$$0 = \frac{\mathrm{d}^2a}{\mathrm{d}t^2}\int_0^t\frac{\partial f_2}{\partial a}\mathrm{d}\tau+\frac{\mathrm{d}a}{\mathrm{d}t}\int_0^t\frac{\partial^2 f_2}{\partial a^2}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}\tau+2\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\partial f_2}{\partial a}.$$

Under the adopted assumption with regard to linearity of the functions and elimination of the integral we get:

$$2\frac{\partial f_2}{\partial a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2-\frac{\mathrm{d}^2a}{\mathrm{d}t^2}f_2[a(t),t] = 0$$

thus, identical equation with the one given above. As the notation (4.11.2) suggested by Kachanov is more convenient it will be applied in further transformations.

For a clearer presentation of the proof the function $p(t)$ was eliminated from equation (4.10.2). In subsequent calculations it seems more convenient, however, to eliminate the function $a(t)$ by means of equation (4.2), which after transformation with regard to $a(t)$ takes the form:

$$a = \frac{M_0+Na_0}{N}\frac{\sum\limits_{i=1}^{\infty}\gamma_i p^{2i-1}}{0,1+\sum\limits_{i=1}^{\infty}\gamma_i p^{2i-1}}. \tag{4.12}$$

Substituting (4.12) into the initial equation (4.10.2) the following relation

$$y(t) = y_0+a_0-y(t)$$

should be applied. The above mentioned relation is easily seen in Fig. 2. Then (4.10.2) adopts the form

$$1-\Psi^{m+1}[y_0+a_0-a(t),t] = A(m+1)\left(\frac{N}{4b_0}\right)^m\int_0^t\frac{[y_0+a_0-a(\tau)+a(\tau)p(\tau)]^{m/n}}{[a^{(n+1)/n}(\tau)\sum\limits_{i=1}^{\infty}\beta_i p^{2i-1}(\tau)]^m}\mathrm{d}\tau. \tag{*}$$

If in the time t the rupture front exceeds the established point of the ordinate y_0, so

$$y(t) = a(t)$$

thus

$$a(t) = y_0+a_0-a(t)$$

and thus

$$y_0+a_0 = 2a(t).$$

Moreover at the moment t which is the rupture moment for the point in consideration, the following takes place:

$$\Psi[a(t), t] = 0.$$

Equation (*) after simple transformations will now be written in the form:

$$1 = A(m+1)\left(\frac{N}{4b_0}\right)^m \int_0^t \frac{\left[p(\tau)-1+2\frac{a(t)}{a(\tau)}\right]^{m/n}}{a^m(\tau)\left[\sum_{i=1}^{\infty} \beta_i p^{2i-1}(\tau)\right]^m} d\tau.$$

In order to fulfil the adopted assumption on linearity of subintegral functions with regard to $a(t)$, $m = n$ must be put in. On one hand, it makes it possible to transform the integral equation into a differential equation, on the other hand, it differs in a small degree from the true ratio of these values. For metals for which values of the material constants are known the ratio m/n oscillates within limits 0·60 and 0·75. Moreover, the assumption $m = n$ has also a physical interpretation. It has been established experimentally that under pure tension the elongation of bars made of the same material is constant at the time of the brittle crack, independent of the value of the stress and the rate of elongation change i.e.

$$t_z \dot{\varepsilon}_x = \text{const.} \tag{4.13}$$

For tension

$$t_z \equiv t_1 = \frac{1}{A(m+1)\sigma_x^m}$$

and according to Norton

$$\dot{\varepsilon}_x = B\sigma_x^n.$$

After substituting one obtains:

$$\frac{B}{A(m+1)} \frac{\sigma_x^n}{\sigma_x^m} = \text{const.} \tag{4.14}$$

The above given equation is true only in case when $m = n$.

Making use of the dependence (4.12) and of the adopted assumption $m = n$ one can write:

$$1 = A(m+1)\left(\frac{N}{4b_0}\right)^m \left(\frac{N}{M_0+Na_0}\right)^m \int_0^t \frac{p(\tau)-1+2\dfrac{\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(t)}{0,1+\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(t)} \dfrac{0,1+\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(\tau)}{\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(\tau)}}{\left[\dfrac{\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(\tau)}{0,1+\sum_{i=1}^{\infty} \gamma_i p^{2i-1}(\tau)}\right]^m \left[\sum_{i=1}^{\infty} \beta_i p^{2i-1}(\tau)\right]^m} d\tau. \tag{4.15}$$

Having twice differentiated (4.15) the following is obtained:

$$\frac{\mathrm{d}^2p}{\mathrm{d}t^2}+[K_1+K_2+K_4+K_5+K_6]\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2=0 \tag{4.16}$$

where K_i denotes:

$$K_1 = \frac{\sum\limits_{i=1}^{\infty}(2i-1)(2i-2)\gamma_i p^{2i-3}}{\sum\limits_{i=1}^{\infty}(2i-1)\gamma_i p^{2i-2}} = \sum_{i=1}^{\infty}\bar{a}_i p^{2i-1}$$

where

$$\bar{a}_1 = 2{\cdot}3\frac{\gamma_2}{\gamma_1},$$

$$\bar{a}_2 = \frac{1}{\gamma_1}(4{\cdot}5\gamma_3-3\bar{a}_1\gamma_2),$$

$$\bar{a}_3 = \frac{1}{\gamma_1}(6{\cdot}7\gamma_4-3\bar{a}_2\gamma_2-5\bar{a}_1\gamma_3),\ldots,$$

and

$$K_2 = -2\frac{0,1\sum\limits_{i=1}^{\infty}(2i-1)\gamma_i p^{2i-2}}{0,1+\sum\limits_{i=1}^{\infty}\gamma_i p^{2i-1}} = \sum_{i=0}^{\infty}\bar{b}_i p^i,$$

where

$$\bar{b}_0 = -2\gamma_1,$$

$$\bar{b}_1 = -\gamma_1\bar{b}_0,$$

$$\bar{b}_2 = -6\gamma_2-\bar{b}_1\gamma_1,$$

$$\bar{b}_3 = -\gamma_1\bar{b}_2-\gamma_2\bar{b}_0,\ldots,$$

and

$$K_3 = \frac{\left(\frac{1}{1+p}-\frac{m}{2}\right)}{\sum\limits_{i=1}^{\infty}\gamma_i p^{2i-1}} = \sum_{i=-1}^{\infty}\bar{c}_i p^i,$$

where

$$\bar{c}_{-1} = \frac{1}{\gamma_1}\left(1-\frac{m}{2}\right),$$

$$\bar{c}_0 = -\frac{1}{\gamma_1},$$

$$\bar{c}_1 = \frac{1}{\gamma_1}(1-\gamma_2\bar{c}_{-1}),$$

$$\bar{c}_2 = -\frac{1}{\gamma_1}(1+\gamma_2\bar{c}_0), \ldots,$$

and

$$K_4 = K_2 K_3 = \sum_{i=-1}^{\infty} \bar{d}_i p^i$$

where

$$\bar{d}_{-1} = \bar{b}_0\bar{c}_{-1} = m-2$$

$$\bar{d}_0 = \bar{c}_{-1}\bar{b}_1 + \bar{c}_0\bar{b}_0$$

$$\bar{d}_i = \sum_{j=0}^{i+1} \bar{b}_j\bar{c}_{i-j}$$

and

$$K_5 = -\frac{1}{1+p} = \sum_{i=0}^{\infty} (-1)^{i+1} p^i$$

$$K_6 = m\frac{\sum_{i=0}^{\infty} (2i-1)\beta_i p^{2i-2}}{\sum_{i=0}^{\infty} \beta_i p^{2i-1}} = mp^{-1} + \sum_{i=1}^{\infty} \bar{e}_i p^{2i-1},$$

where

$$\bar{e}_1 = 2m\frac{\beta_1}{\beta_2},$$

$$\bar{e}_2 = 2m(2\beta_3 - \beta_2^2),$$

$$\bar{e}_3 = 6m(\beta_4 - \beta_2\bar{e}_2 - \beta_3\bar{e}_1), \ldots .$$

Denoting now

$$K = K_1 + K_2 + K_4 + K_5 + K_6 = \sum_{i=-1}^{\infty} R_i p^i$$

where

$$R_{-1} = 2(m-1),\ R_0 = \bar{b}_0 + \bar{d}_0,\ R_1 = 1 + \bar{a}_1 + \bar{b}_1 + \bar{d}_1 + \bar{e}_1, \ldots$$

the equation (4.16) will be written in form

$$\frac{d^2p}{dt^2}+\left(\sum_{i=-1}^{\infty} R_i p^i\right)\left(\frac{dp}{dt}\right)^2 = 0. \tag{4.17}$$

Initial conditions will have the forms:
(a) for $t = t_1$

$$p(t_1) \stackrel{\text{df}}{=} p_0 = \sum_{i=1}^{\infty} v_i \delta_0^{2i-1} \tag{4.18.1}$$

(b) for $t = t_1$ dp/dt is calculated from formula (4.15) after differentiation, viz.

$$\left.\frac{dp}{dt}\right|_{t=t_1} \stackrel{\text{df}}{=} p_0' = -\frac{1+p_0}{2t_1}\,\frac{\left[\sum_{i=1}^{\infty} \gamma_i p_0^{2i-1}\right]\left[0,1+\sum_{i=1}^{\infty} \gamma_i p_0^{2i-1}\right]}{0,1\sum_{i=1}^{\infty}(2i-1)\gamma_i p_0^{2i-2}}. \tag{4.18.2}$$

Determining

$$\bar{A} = (1+p_0)\sum_{i=1}^{\infty} \gamma_i p_0^{2i-1}$$

and

$$\bar{B} = \frac{0,1+\sum_{i=1}^{\infty} \gamma_i p_0^{2i-1}}{0,1\sum_{i=1}^{\infty}(2i-1)\gamma_i p_0^{2i-2}} = \sum_{i=0}^{\infty} \bar{f}_i p_0^i$$

where

$$\bar{f}_0 = \frac{1}{\gamma_1}, \qquad \bar{f}_1 = 1, \qquad \bar{f}_2 = -3\frac{\gamma_2}{\gamma_1}, \qquad \bar{f}_3 = -2\frac{\gamma_2}{\gamma_1}, \ldots$$

and performing subsequent multiplication

$$\bar{A}\times\bar{B} = \sum_{i=1}^{\infty} \bar{g}_i p_0^i$$

where

$$\bar{g}_1 = 1, \qquad \bar{g}_2 = 1+\gamma_1, \qquad \bar{g}_3 = \gamma_1(\bar{f}_1+\bar{f}_2)+\gamma_2\bar{f}_0, \ldots$$

the condition (4.18.2) is presented in form:

$$p_0' = -\frac{1}{2t_1}\sum_{i=1}^{\infty} \bar{g}_i p_0^i. \tag{4.18.2a}$$

For fibres under compression at $t = 0$ the integral in formula (4.15) should be taken within the limits t^*, t, where t^* denotes the time when the neutral axis reaches the considered point. It can easily be shown, that then too, the integral equation can be transformed—in the manner discussed above—into the differential equation (4.17). It should be stressed, however, that for these fibres the material constants would be functions of the variable y and time t, as effect of pressure. Application of equation (2.2) is, thus, in this period a certain approximation of the solution of the discussed problem.

The integral of equation (4.17) is found by substituting

$$\frac{dp}{dt} = F(p), \qquad \frac{d^2p}{dt^2} = \frac{dF}{dp}F(p).$$

The equation will take the form:

$$\frac{dF}{dp} + \left(\sum_{i=-1}^{\infty} R_i p^i\right) F = 0.$$

Separation of variables and bilateral integration gives:

$$F = D_1 \frac{1}{p^{R_{-1}}} \exp\left(-\sum_{i=0}^{\infty} \frac{R_i}{i+1} p^{i+1}\right), \tag{4.19}$$

where the integration constant D_1 is calculated from the condition that for $t = t_1$ we have $p = p_0$ and $F = p_0'$. Thus

$$D_1 = p_0' p_0^{R_{-1}} \exp\left(\sum_{i=0}^{\infty} \frac{R_i}{i+1} p_0^{i+1}\right).$$

By developing the exponential function into power series and making use of (4.18.2a) the following is obtained:

$$D_1 = \frac{1}{2t_1}\left[\sum_{i=1}^{\infty} \bar{g}_i p_0^i\right]\left[\sum_{i=0}^{\infty} S_i p_0^i\right] p_0^{R_{-1}}$$

where

$$S_0 = 1, \qquad S_1 = R_0, \qquad S_2 = \frac{1}{2}R_1 + \frac{1}{2!}R_0^2, \qquad S_3 = \frac{1}{3}R_2 + \frac{1}{2!}2R_0\frac{R_1}{2} + \frac{1}{3!}R_0^3, \ldots .$$

Multiplication of the series gives

$$D_1 = -\frac{p_0^{R_{-1}+1}}{2t_1} \sum_{i=0}^{\infty} U_i p_0^i \tag{4.20}$$

where

$$U_0 = 1, \qquad U_1 = S_1\bar{g}_1 + S_0\bar{g}_2, \ldots$$

$$U_i = \sum_{j=1}^{i+1} \bar{g}_j S_{i-j+1} \quad \text{for } i \geq 1.$$

Introducing the former variable into (4.19) and separating the variables, one gets subsequently:

$$p^{R_{-1}} \exp\left(\sum_{i=0}^{\infty} \frac{R_i}{i+1} p^{i+1}\right) dp = D_1\, dt.$$

Making use of the development of the exponential function into a power series and integrating it bilaterally we have:

$$\sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p_i^{i+R_{-1}+1} = D_1 t + D_2. \tag{4.21}$$

D_2 is determined from the conditions (4.18)

$$D_2 = \sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p_0^{i+R_{-1}+1} - D_1 t_1. \tag{4.22}$$

Thus

$$D_1 t = \sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p^{i+R_{-1}+1} - \sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p_0^{i+R_{-1}+1} + D_1 t_1$$

so

$$\frac{t}{t_1} = 2\frac{\sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p_0^i}{\sum_{i=0}^{\infty} U_i p_0^i} - 2\frac{\sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p^{i+R_{-1}+1}}{p_0^{R_{-1}+1} \sum_{i=0}^{\infty} U_i p_0^i} + 1. \tag{4.23}$$

The rupture time $t = t_z$ is obtained for $a \to 0$, thus, for $p \to 0$ as well [on the grounds of (4.12)]

$$\frac{t_z}{t_1} = 1 + 2\frac{\sum_{i=0}^{\infty} \frac{S_i}{i+R_{-1}+1} p_0^i}{\sum_{i=0}^{\infty} U_i p_0^i}. \tag{4.24}$$

5. SMALL ECCENTRICITY

5.1. *Stress distribution*

Acting as in point 4 internal forces can be expressed

$$\begin{aligned} N &= 4ab_0\sigma_c\left[1 + \sum_{i=1}^{\infty} \mathrm{d}_i q^{2i}\right] \\ M &= 4a^2 b_0 \sigma_c \sum_{i=0}^{\infty} e_i q^{2i-1} \end{aligned} \tag{5.1}$$

where

$$\mathrm{d}_i = \frac{1}{(2i+1)!n^{2i}} \prod_{j=1}^{2i-1} (1-jn)$$

$$e_i = \frac{1}{(2i+1)!n^{2i-1}} \prod_{j=1}^{2i-2} (1-jn).$$

Forming a quotient $(10M)/(aN)$ and performing the division of series one obtains

$$\Delta(a) \stackrel{\mathrm{df}}{=} \frac{10M}{aN} = \frac{10[M_0 + N(a_0 - a)]}{aN} = \sum_{i=1}^{\infty} f_i q^{2i-1}, \tag{5.2}$$

where

$$f_i = 10\left[e_i - \sum_{j=1}^{i-1} a_j f_{i-j}\right],$$

and

$$a_i = \frac{1}{i!n^i} \prod_{j=1}^{i-1} (1-jn).$$

Inversion of the above quoted series gives the required function

$$q = \sum_{i=1}^{\infty} g_i \Delta^{2i-1} \tag{5.3}$$

where

$$g_1 = \frac{1}{f_1}, \qquad g_2 = -\frac{f_2}{f_1^4}, \qquad g_3 = \frac{3f_2^2}{f_1^7} - \frac{f_3}{f_1^6}, \ldots .$$

Now, the stress distribution can be determined as

$$\sigma_x = \frac{N}{4ab_0\left[1 + \sum_{i=1}^{\infty} d_i q^{2i}\right]}(1+q\eta)^{1/n}, \tag{5.4.1}$$

or

$$\sigma_x = \frac{M}{4a^2b_0 \sum_{i=1}^{\infty} e_i q^{2i-1}}(1+q\eta)^{1/n}. \tag{5.4.2}$$

The above mentioned formulae are valid when $q \leq 1$ i.e. when $\Delta(a) < \Delta_{max}$.

For $0 \leq t < t_1$ (where t_1 denotes the time of arising of the first cracks) the maximum value Δ_0 will be for $q = 1$, thus

$$\Delta_{0max} = \sum_{i=1}^{\infty} f_i. \tag{5.5}$$

For $t > t_1$ cracks of external fibres will appear, the height of the cross section will decrease and the load will change: under constant axial force the bending moment will increase from the value M_0 to $M = M_0 + N(a_0 - a)$, thus, the eccentricity Δ will increase and so will the value q. Convergence of the applied series will be secured if $q \leq 1$, thus, the minimum value $a = a^*$ (for which stress on the whole section will still be of the same sign), is calculated from (5.2) inserting $q = 1$

$$\Delta_{0max} = \frac{10[M_0 + N(a_0 - a^*)]}{Na^*}$$

where from

$$a^* = \frac{M_0 + a_0 N}{N(1 + 0,1\Delta_{0max})}. \tag{5.6}$$

In calculating the rupture time three stages must be distinguished:

(a) time of arising of first cracks—t_1
(b) time of rupture of the cross section to height a^*—t_2
(c) time of rupture of the whole section—t_z

5.2. *Time of arising of first cracks*

For $0 \leq t < t_1$ we have $a = a_0$, $q = q_0$, $y = y_0$, and the normal stress does not depend on time.

Time t_1 is determined by the formula (2.7) which after using (5.4) takes the form:

$$t_1 = \frac{\left\{4a_0 b_0\left[1+\sum_{i=1}^{\infty} d_i q_0^{2i}\right]\right\}^m}{A(m+1)N^m(1+q_0)^{m/n}} \tag{5.7.1}$$

or

$$t_1 = \frac{\left\{4a_0^2 b_0 \sum_{i=1}^{\infty} e_i q^{2i-1}\right\}^m}{A(m+1)M_0^m(1+q_0)^{m/n}}. \tag{5.7.2}$$

When $M_0 = 0$ then $q_0 = 0$; thus, we obtain:

$$t_{1N} = \frac{\{4a_0 b_0\}^m}{A(m+1)N^m},$$

which is a known formula for the time of rupture of a bar under axial tension.

Distribution of continuity at $t = t_1$ is given by the formula

$$\Psi^{m+1}(y, t_1) = 1 - \left\{\frac{1+q\eta}{1+q_0}\right\}^m. \tag{5.8}$$

5.3. *Rupture time of the section to the height a^**

In the time interval $(0, t_2)$ the integral of equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -A\left(\frac{\sigma_x}{\Psi}\right)^m$$

takes the form

$$1 - \Psi^{m+1}(y, t) = A(m+1)\left(\frac{N}{4b_0}\right)^m \int_0^t \frac{(1+q\eta)^{m/n}}{a^m\left[1+\sum_{i=1}^{\infty} d_i q^{2i}\right]^m}\,\mathrm{d}\tau. \tag{5.9}$$

Making use of this relation and considering the fact that

$$2a(t) = y_0 + a_0,$$

and

$$\Psi[a(t), t] = 0$$

when the rupture front reaches the established point of the ordinate y_0, the equation (5.9) can be written in the form

$$1 = A(m+1)\left(\frac{N}{4b_0}\right)^m \int_0^t \frac{\left\{1+q(\tau)\left[2\frac{a(t)}{a(\tau)}-1\right]\right\}^{m/n}}{a^m(\tau)\left\{1+\sum_{i=1}^{\infty} d_i q^{2i}(\tau)\right\}^m}\,\mathrm{d}\tau. \tag{5.10}$$

From formula (5.2) a will be expressed in terms of q:

$$a = \frac{M_0 + Na_0}{N} \frac{1}{1+0,1 \sum_{i=1}^{\infty} f_i q^{2i-1}}. \tag{5.11}$$

After introducing (5.11) into (5.10) and after some simple transformations we obtain

$$1 = A(m+1)\left(\frac{N}{4b_0}\right)^m \left(\frac{N}{M_0+Na_0}\right)^m \int_0^t \frac{\left\{1+q(\tau)\left[2\dfrac{1+0,1\sum_{i=1}^{\infty} f_i q^{2i-1}(t)}{1+0,1\sum_{i=1}^{\infty} f_i q^{2i-1}(\tau)} - 1\right]\right\}^{m/n}}{\left\{\dfrac{1+\sum_{i=1}^{\infty} d_i q^{2i}(\tau)}{1+0,1\sum_{i=1}^{\infty} f_i q^{2i-1}(\tau)}\right\}^m} \mathrm{d}\tau. \tag{5.12}$$

Next we proceed in an analogous way as for large eccentricity: (5.12) is twice differentiated with regard to time, and

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + [L_1 + L_2 L_3 + L_4 + L_5]\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 = 0 \tag{5.13}$$

where

$$L_1 = \frac{\sum_{i=1}^{\infty} (2i-1)(2i-2) f_i q^{2i-3}}{\sum_{i=1}^{\infty} (2i-1) f_i q^{2i-2}}, \qquad L_2 = -\frac{2+m(1+q)}{1+q}$$

$$L_3 = \frac{0,1 \sum_{i=1}^{\infty} (2i-1) f_i q^{2i-2}}{1+0,1\sum_{i=1}^{\infty} f_i q^{2i-1}}, \qquad L_4 = -\frac{1}{1+q}$$

$$L_5 = m\frac{\sum_{i=1}^{\infty} 2i d_i q^{2i-1}}{1+\sum_{i=1}^{\infty} d_i q^{2i}}$$

is obtained.

Performing multiplications and divisions of the series in brackets one obtains:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \left(\sum_{i=0}^{\infty} r_i q^i\right)\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 = 0. \tag{5.14}$$

Initial conditions are

(a) for $t = t_1$

$$q(t_1) \stackrel{\mathrm{df}}{=} q_0 = \sum_{i=1}^{\infty} g_i \Delta_0^{2i-1} \tag{5.15.1}$$

(b) for $t = t_1$ the value $\mathrm{d}q/\mathrm{d}t$ is calculated by use of the first derivate of equation (5.12)

$$\left.\frac{\mathrm{d}q}{\mathrm{d}t}\right|_{t=t_1} \overset{\mathrm{df}}{=} q_0' = \frac{1}{2t_1}\frac{1+q_0}{q_0}\frac{1+0,1\sum_{i=1}^{\infty} f_i q_0^{2i-1}}{0,1\sum_{i=1}^{\infty}(2i-1)f_i q_0^{2i-2}}. \tag{5.15.2}$$

The integral of equation (5.14) will be found substituting

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \vartheta(q) \rightarrow \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{\mathrm{d}\vartheta}{\mathrm{d}q}\vartheta(q).$$

This will permit to write (5.14) in form:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}q} + \left(\sum_{i=0}^{\infty} r_i q^i\right)\vartheta = 0. \tag{5.16}$$

Separation of variables and a bilateral integration gives:

$$\vartheta = C_1 \exp\left(-\sum_{i=0}^{\infty}\frac{r_i}{i+1}q^{i+1}\right). \tag{5.17}$$

Making use of condition (5.12.2) one obtains:

$$C_1 = q_0' \exp\left(\sum_{i=0}^{\infty}\frac{r_i}{i+1}q_0^{i+1}\right).$$

Returning to the former variable and separating the variables

$$\mathrm{d}t = \mathrm{d}q C_1^{-1}\exp\left(\sum_{i=0}^{\infty}\frac{r_i}{i+1}q^{i+1}\right)$$

will be obtained from (5.17). After a bilateral integration we obtain

$$C_1^{-1}\sum_{i=0}^{\infty}\frac{s_i}{i+1}q^{i+1} = t + C_2 \tag{5.18}$$

where

$$s_0 = 1, \quad s_1 = r_0, \qquad s_2 = \frac{1}{2}r_1 + \frac{1}{2!}r_0^2, \ldots .$$

The constant C_2 is determined from condition (5.15.1)

$$C_2 = C_1^{-1}\sum_{i=0}^{\infty}\frac{s_i}{i+1}q_0^{i+1} - t_1.$$

Finally

$$t = t_1 + C_1^{-1}\sum_{i=0}^{\infty}\frac{s_i}{i+1}(q^{i+1} - q_0^{i+1}) \tag{5.19}$$

is obtained.

Damage will reach fibres $a = a^*$ when $q = 1$. Thus,

$$t_2 = t_1 + C_1^{-1} \sum_{i=0}^{\infty} \frac{s_i}{i+1}(1 - q_0^{i+1}) \tag{5.20}$$

or

$$\frac{t_2}{t_1} = 1 + \frac{1}{t_1 C_1} \sum_{i=0}^{\infty} \frac{s_i}{i+1}(1 - q_0^{i+1}) \tag{5.20.1}$$

For the limiting case $q_0 = 1$ it is evident that

$$\frac{t_2}{t_1} = 1,$$

because condition $q_0 = 1$ corresponds to such a ratio M/N which gives a vanishing stress (for $0 \leq t < t_1$) on the lower edge. Thus, for $t = t_1 + \mathrm{d}t$ the stress will change sign and a large eccentricity will be dealt with. Moreover, because of $\Delta_0 = (10M_0)/(a_0 N) = \Delta_{\max}$ we then have:

$$a^* = \frac{M_0 + Na_0}{N\left(1 + \dfrac{M_0}{a_0 N}\right)} = \frac{Na_0\left(1 + \dfrac{M_0}{a_0 N}\right)}{N\left(1 + \dfrac{M_0}{a_0 N}\right)} = a_0.$$

5.4. *Time of total destruction of the cross section*

For time $t > t_2$ compressive as well as tension stresses appear, and a large eccentricity must thus be dealt with. The phenomenon of propagation of the rupture front will be described by the differential equation (4.17), but the initial conditions, which must be given for $t = t_2$, will differ from the conditions (4.18), For $t = t_2$ we have:

(a) $p(t_2) = 1$ (5.21.1)

(b) $\left.\dfrac{\mathrm{d}p}{\mathrm{d}t}\right|_{t=t_2} = -\left.\dfrac{\mathrm{d}q}{\mathrm{d}t}\right|_{t=t_2}.$ (5.21.2)

It follows from equation (5.18) that

$$\sum_{i=0}^{\infty} \frac{s_i}{i+1} q^{i+1} = C_1 t + C_1 C_2.$$

Bilateral differentiation of this dependence with regard to time gives

$$\frac{\mathrm{d}q}{\mathrm{d}t} \sum_{i=0}^{\infty} s_i q^i = C_1,$$

from where

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{C_1}{\sum_{i=0}^{\infty} s_i q^i}.$$

Substituting here

$$C_1 = \frac{1}{2t_1}\frac{1+q_0}{q_0}\frac{1+0,1\sum_{i=1}^{\infty} f_i q_0^{2i-1}}{0,1\sum_{i=1}^{\infty} f_i q_0^{2i-2}(2i-1)}\sum_{i=0}^{\infty} s_i q_0^i$$

for C_1 the following will be finally obtained:

$$\left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{t=t_2} = -\frac{1}{2t_1}\frac{1+q_0}{q_0}\frac{1+0,1\sum_{i=1}^{\infty} f_i q_0^{2i-1}}{0,1\sum_{i=1}^{\infty}(2i-1)f_i q_0^{2i-2}}\frac{\sum_{i=0}^{\infty} s_i q^i}{\sum_{i=0}^{\infty} s_i} \tag{5.22}$$

(as for $t = t_2$ $q = 1$).

The equation (4.17) is solved in an analogous way as in Section 4. Introducing the function $G(p) = \mathrm{d}p/\mathrm{d}t$, and performing an integration

$$G(p) = E_1\frac{1}{p^{R_{-1}}}\exp\left(-\sum_{i=0}^{\infty}\frac{R_i}{i+1}p^{i+1}\right) \tag{5.23}$$

is obtained.

For

$$t = t_2, \qquad p = 1, \qquad F = \left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{t=t_2},$$

where from

$$E_1 = -C_1\frac{\sum_{i=0}^{\infty} S_i}{\sum_{i=0}^{\infty} s_i}. \tag{5.24}$$

Integration of equation (5.23) gives

$$\sum_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}p^{i+R_{-1}+1} = E_1 t + E_2.$$

The constant E_2 is determined by use of condition (5.21.1):

$$E_2 = \sum_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1} - E_1 t_2.$$

Thus

$$t = \frac{1}{E_1}\sum_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}(p^{i+R_{-1}+1}-1)+t_2.$$

For $t = t_z$ $p = 0$ and

$$\frac{t_z}{t_2} = 1+\frac{\sum_{i=0}^{\infty} s_i}{t_2 C_1\sum_{i=0}^{\infty} S_i}\sum_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}. \tag{5.25}$$

Denoting

$$C_1 = \frac{1}{2t_1}\bar{C}_1$$

$$\frac{t_z}{t_2} = 1+2\frac{t_1}{t_2}\frac{\left[\sum\limits_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}\right]\left[\sum\limits_{i=0}^{\infty}s_i\right]}{\bar{C}_1\sum\limits_{i=0}^{\infty}S_i} \tag{5.26.1}$$

or

$$\frac{t_z}{t_2} = \frac{t_2}{t_1}+\frac{2}{\bar{C}_1}\frac{\sum\limits_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}}{\sum\limits_{i=0}^{\infty}S_i}\sum_{i=0}^{\infty}s_i \tag{5.26.2}$$

is obtained.

Making use of equation (5.20) one obtains:

$$\frac{t_z}{t_1} = 1+\frac{2}{\bar{C}_1}\left[\sum_{i=0}^{\infty}\frac{s_i}{i+1}(1-q_0^{i+1})+\frac{\sum\limits_{i=0}^{\infty}\frac{S_i}{i+R_{-1}+1}}{\sum\limits_{i=0}^{\infty}S_i}\sum_{i=0}^{\infty}s_i\right]. \tag{5.27}$$

6. NUMERICAL ANALYSIS

The ratio of the total time of destruction of the rectangular cross section to the time of arising of first cracks is determined by the formulae (4.24) and (5.27)

$$\frac{t_z}{t_1} = \begin{cases} 1+2\dfrac{\sum\limits_{i=0}^{\infty}\dfrac{S_i}{i+R_{-1}+1}p_0^i}{\sum\limits_{i=0}^{\infty}U_i p_0^i} & \text{for } \Delta > \Delta_{\min} \\[2ex] 1+2\dfrac{1}{\bar{C}_1}\left[\sum\limits_{i=0}^{\infty}\dfrac{s_i}{i+1}(1-q_0^{i+1})+\dfrac{\sum\limits_{i=0}^{\infty}\dfrac{S_i}{i+R_{-1}+1}}{\sum\limits_{i=0}^{\infty}S_i}\sum\limits_{i=0}^{\infty}s_i\right] & \text{for } \Delta < \Delta_{\max}. \end{cases}$$

As has already been mentioned in the introduction, the time of arising of the first cracks is often the adopted criterion of destruction. Evaluation of numerical values of terms describing this process in the above given equations from the moment when the first cracks arise to that of total destruction is now of great interest. These terms are functions of two parameters: material constant $m = n$ (all series coefficients are expressed by this parameter) and initial loads M_0 and N_0, expressed by q_0, and p_0, respectively.

For pure tension ($q_0 = 0$) we have

$$\lim_{q_0 \downarrow 0} \frac{1}{\bar{C}_1} = \lim_{q_0 \downarrow 0} \frac{q_0}{1+q_0} \frac{0,1 \sum_{i=1}^{\infty} (2i-1) f_i q_0^{2i-2}}{1+0,1 \sum_{i=1}^{\infty} f_i q_0^{2i-1}} \sum_{i=0}^{\infty} S_i q_0^i = 0$$

and thus

$$\left.\frac{t_z}{t_1}\right|_{q_0=0} = 1, \tag{6.1}$$

which means that under tension the time of arising of the first cracks (independent of m and therefore of temperature) is the same as the time of total destruction of the whole cross section.

For pure bending, $p_0 = 0$, the ratio t_z/t_1 is determined by the formula (4.24):

$$\left.\frac{t_z}{t_1}\right|_{p_0=0} = 1+\frac{2}{2m-1}. \tag{6.2}$$

As can be seen, this ratio depends on the material constant m and reaches an upper limit when $m \to 1$:

$$\max \left.\frac{t_z}{t_1}\right|_{p_0=0} = 3.$$

The diagram t_z/t_1 is shown in Fig. 3. Calculations were restricted to at most five terms of the individual series. An approximation was applied in calculating the series K_2, because of its divergence for $p_0 > 0{\cdot}3$, by means of Newton's formula, wherein the polynomial terms of the fifth degree were maintained.

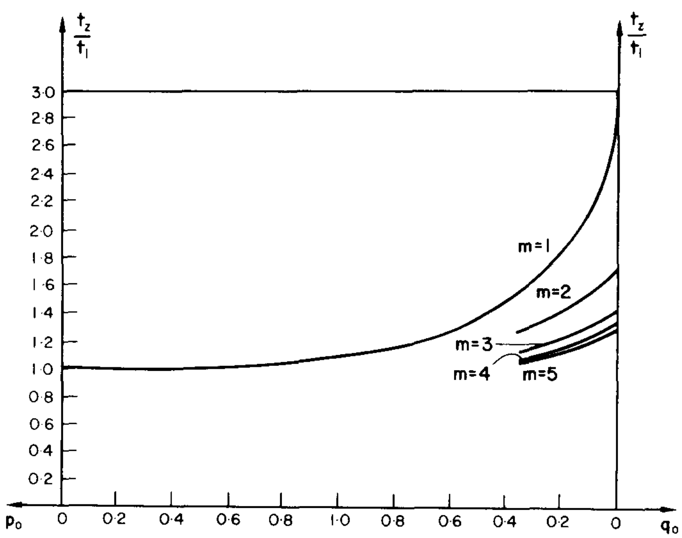

FIG. 3.

7. CONCLUSIONS

The analytical and numerical calculations which were carried out under the assumption $m = n$ permit to advance two basic conclusions.

(1) The ratio of time of total destruction to the time of arising of the first cracks depends on the ratio of the bending moment and the normal force at $t = 0$, on temperature and on the kind of material (constants m and n). This ratio lies generally close to one; only for ratios M to N close to the case of pure bending (large M) and for material constant value close to $m = 1$, is this ratio considerably greater than one.
(2) The calculations are complicated even for a simple shape of the cross section as a rectangular one, and considerable difficulties are encountered. The use of a computer program (for an arbitrary shape of the cross section) would simplify the calculations.

In view of the foregoing it seems that in engineering calculations the time t_1 of the appearance of the first cracks can be adopted as the criterion for a safe life span of the structure.

The fact that the assumption of stationary creep up to the time of total destruction caused a too high value of the above calculated ratio t_z/t_1 is an advantageous circumstance. When nonstationary creep is taken into account this ratio would be still closer to unity, even for pure bending.

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Абстракт—Рассматривается задача полного разрушения балки прямоугольного поперечного сечения, под влиянием момента изгиба и осевой растягивающей силы. Задача основана на двух фундаментальных законах: законе ползучести Одквиста и законе хрупкого разрушения Качанова. Используется метод степенных рядов для решения основного интегрального уравнения и для выполнения расчетов.